Math 371 **Name:** ________________________
Spring 2019
Practice exam
02/19/2019
Time Limit: 80 Minutes ID ________________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ________________________

This exam contains 11 pages (including this cover page) and 10 questions.
Total of points is 100.

- Check your exam to make sure all 11 pages are present.
- You may use writing implements and a single handwritten sheet of 8.5"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) State the definition of an operation of group G on a set S. State the property for the operation to be transitive.

Definition of operation:

a map $G \times S \to S$.

$(g, s) \mapsto g \cdot s$.

such that ① $(g_1 g_2) \cdot s = g_1 (g_2 s)$

$\forall g_1, g_2 \in G, s \in S$.

② $1 \cdot s = s$. $\forall s \in S$.

Transitive means the operation has only one orbit.

2. (10 points) Write the element $(123)(234) \in S_4$ as product of disjoint cycles.

1 2 3 4
1 3 4 2
2 1 4 3

So $(123)(234) = (12)(34)$

3. (10 points) Find the Sylow 2-subgroup of S_4.

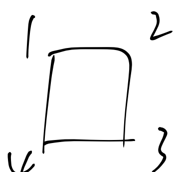
$$|S_4| = 4\times 3\times 2\times 1 = 8\times 3$$

Sylow 2-subgroups are subgroups of order 8.

number of Sylow 2-subgroups $s = 1$ or 3.

Since every transposition (i,j) $i\neq j$ is contained in some sylow 2-subgroup. and all the (i,j) generate S_4. $s\neq 1$, $s=3$.

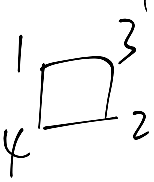
Consider the action of D_4 on the vertices of a square, $D_4 \to S_4$ as a subgroup of S_4



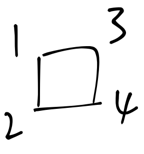
$H_1 = D_4 = \{1, (1234), (13)(24), (1432), (12)(34), (13), (24), (14)(23)\}$

Consider the conjugation of D_4 in S_4

(or reindex the vertices).



$H_2 = \{1. (1324), (12)(34). (1423), (13)(24). (12), (34), (14)(23)\}$.



$H_3 = \{1, (1342), (14)(23), (1243), (13)(24). (14), (23), (12)(34)\}$.

4. (10 points) Find all the normal subgroups of D_6.

$$D_6 = \left\{ \begin{matrix} 1, & x, & x^2, & x^3, & x^4, & x^5 \\ y, & xy, & x^2y, & x^3y, & x^4y, & x^5y \end{matrix} \right\}$$

The conjugacy classes of D_6 are

$\underset{1}{\{1\}}$, $\underset{2}{\{x, x^5\}}$, $\underset{2}{\{x^2, x^4\}}$, $\underset{1}{\{x^3\}}$.

$\underset{3}{\{y, x^2y, x^4y\}}$, $\underset{3}{\{xy, x^3y, x^5y\}}$.

$|D_6| = 12$, the normal subgroup is the union of conjugacy classes containing $\{1\}$ whose order divides 12,

so 1, $\{1\}$

$1+1$ $\{1, x^3\}$ not a subgroup

$1+2$, $\cancel{\{1, x, x^5\}}$ $\{1, x^2, x^4\}$

$1+3$, $\cancel{\{1, y, x^2y, x^4y\}}$, $\cancel{\{1, xy, x^3y, x^5y\}}$. not subgroups.

$1+2+2+1$, $\{1, x, x^2, x^3, x^4, x^5\}$

$1+2+3$, $\{1, x^2, x^4, y, x^2y, x^4y\}$.

$1+1+2$ $\cancel{\{1, x, x^5, x^3y\}}$ $\cancel{\{1, x^2, x^4, x^3y\}}$

not subgroups.

$1+1+2+2+3+3$. D_4.

6 normal subgroups.

$\{1\}$, D_4,

$\{1, x^3y\}$, $\{1, x^2, x^4y\}$

$\{1, x, x^2, x^3, x^4, x^5\}$

$\{1, x^2, x^4, y, x^2y, x^4y\}$.

5. (10 points) Classify all finite groups of order 45.

$|G| = 45 = 5 \times 9$.

number of Sylow 3-subgroup $s \mid 5$ and

$s \equiv 1 \pmod 3$, so $s = 1$.

number of Sylow 5-subgroup $s' \mid 3$ and

$s' \equiv 1 \pmod 5$ so $s' = 1$.

Let H be the unique sylow 3-subgroup.

K be the unique Sylow 5-subgroup.

Then H, K are normal subgroups of G.

$H \cap K = \{1\}$, $HK = G$.

$|H| = 9$, so $H \cong C_3 \times C_3$, or C_9.

$G \cong C_3 \times C_3 \times C_5$ or $C_9 \times C_5$.

6. (10 points) Classify all finite groups of order 10.

$|G| = 10$,

number of Sylow 5-group is 1.

Let H be the Sylow 5-group.

H is a normal subgroup of G.

Let K be a Sylow 2-subgroup.

Then $HK = G$. $H \cap K = \{1\}$.

$G \cong H \rtimes_\varphi K$, with

$\varphi : K \to \mathrm{Aut}\, H$.

$k \mapsto \varphi(k) : h \mapsto khk^{-1}$.

Let $H = \langle x \rangle$. $x^5 = 1$.

$K = \langle y \rangle$. $y^2 = 1$.

$yxy^{-1} = x^j$. then $j^2 \equiv 1 \pmod 5$.

so $j \equiv 1$. or $4 \pmod 5$. $yxy^{-1} = x$. or x^4.

$G \cong C_2 \times C_5$ or D_5

7. (10 points) Prove that a group of order 200 is not a simple group.

$|G| = 200 = 2^3 \times 5^2$

number of Sylow 5-groups s

$s \mid 8$, $s \equiv 1 \pmod 5$

so $s = 1$.

So Sylow 5-group is unique,

and it is a nontrivial normal subgroup,

So G is not simple.

8. (10 points) Prove that $SO(2)$ is isomorphic to $\mathbb{R}/\mathbb{Z}$.

$$\varphi : \mathbb{R} \longrightarrow SO(2).$$

$$\theta \longmapsto \begin{bmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}$$

is a group homomorphism.

$$\ker\varphi = \mathbb{Z}$$

So $SO(2) \cong \mathbb{R}/\mathbb{Z}$

9. (10 points) The operation of finite group G on set S is transitive and H is a normal subgroup of G. Prove that the orbits under the operation of H on S have the same number of elements.

Let O_1. O_2 be two orbits of the operation of H on S.

Let $x \in O_1$, $y \in O_2$, then $y = gx$ for some $g \in G$ (because G operation is transitive)

The stabilizer of x under H action is

$H_x = H \cap G_x = \{ g \in H \mid gx = x \}$.

$H_y = H \cap G_y$. $G_y = g G_x g^{-1}$

So $H \cap G_y = (gHg^{-1}) \cap g G_x g^{-1}$

$= g (H \cap G_x) g^{-1}$

$= g H_x g^{-1}$

So $|H_x| = |H_y|$. $|O_1| = \frac{|H|}{|H_x|} = \frac{|H|}{|H_y|} = |O_2|$

10. (10 points) Let p be a prime number. Prove the center $Z(G)$ of a nonabelian group G of order p^3 must have order p.

G is a p-group. So $Z(G)$ is nontrivial

$G/Z(G)$ is also a p-group.

Since G is nonabelian, $Z(G) \neq G$.

so $|Z(G)| = p$ or p^2.

If $|Z(G)| = p^2$, then $|G/Z(G)| = p$

$G/Z(G)$ is a cyclic group.

Let $G/Z(G) = \langle x \cdot Z(G) \rangle$.

then any element in $G/Z(G)$ has the form $x^i y$, $y \in Z(G)$.

$(x^i y_1)(x^j y_2) = x^i x^j y_1 y_2 = (x^j y_2)(x^i y_1)$

$\forall y_1, y_2 \in Z(G)$, so G is abelian.

Contradiction! so $|Z(G)| = p$.